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A stationary cylindrically symmetric electrovac space–time

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Abstract. A stationary cylindrically symmetric electrovac solution of the Einstein–Maxwell equations is derived in which the electromagnetic field is null. The resulting space–time contains no time-like hypersurface-orthogonal Killing fields so that it is non-static.

1. Introduction

Despite the existence of well known *bona fide* (i.e. non-static) stationary axially symmetric electrovac solutions to the Einstein–Maxwell equations, it is difficult to find such solutions for the case of cylindrical symmetry in the literature. The only example in the exhaustive survey of Kramer *et al* (1980) is that due to Wilson (1968). However, a careful check of the latter shows that it is not in fact an electrovac solution (this has subsequently been checked by M MacCallum who comes to the same conclusion). The stationary cylindrical solutions of Arbex and Som (1973) correspond to taking $w = \text{constant}$ in (2.1) below and, as noted by the authors themselves, are simply static fields viewed from a rotating coordinate system.

In the present paper we exhibit a stationary cylindrically symmetric electrovac space–time that has no hypersurface-orthogonal time-like Killing fields and is therefore non-static. The integration of the Einstein–Maxwell equations is facilitated by taking the electromagnetic field to be null. In §2 the metric is derived and in §3 the properties of the resulting space–time are discussed.

2. The metric

The metric of a stationary cylindrically symmetric electrovac space–time may be written in the form

$$ds^2 = -f(dt + w\,d\phi)^2 + f^{-1}[r^2\,d\phi^2 + e^{2v}(dz^2 + dr^2)] \tag{2.1}$$

where (ϕ, z, r) are cylindrical coordinates and f, w and v are functions of r only. The only non-zero components of the electromagnetic field tensor with respect to the obvious orthonormal basis

$$\theta^0 = f^{1/2}(dt + w\,d\phi) \qquad \theta^1 = f^{-1/2} r\,d\phi$$
$$\theta^2 = f^{-1/2} e^v\,dz \qquad \theta^3 = f^{-1/2} e^v\,dr \tag{2.2}$$

are $F_{03} = -F_{30}$ and $F_{13} = -F_{31}$. The Einstein–Maxwell equations are

$$dF = 0 \qquad d^*F = 0 \tag{2.3}$$

$$R_{ab} = -\kappa E_{ab} \tag{2.4}$$

where

$$F = F_{ab}\theta^a \wedge \theta^b \qquad {}^*F = \tfrac{1}{2}\eta_{abcd}F^{cd}\theta^a \wedge \theta^b \tag{2.5}$$

and

$$E_{ab} = 2\kappa^{-1}(F_a{}^cF_{bc} - \tfrac{1}{4}\eta_{ab}F_{cd}F^{cd}) \qquad \kappa = 8\pi. \tag{2.6}$$

The indices refer to the orthonormal basis throughout.

We seek solutions of these equations for which $F_{03} = F_{13} = u(r)$ (say). This means that the electromagnetic field is null with $k^a = (1, -1, 0, 0)$ as the degenerate principal null direction and the only non-zero components of E_{ab} are E_{00}, E_{01} and E_{11} with

$$\kappa E_{00} = \kappa E_{01} = \kappa E_{11} = 2u^2. \tag{2.7}$$

The equations (2.4) reduce to

$$r^2ff'' - r^2f'^2 + rff' + f^4w'^2 = 4r^2fu^2\,e^{2v} \tag{2.8}$$

$$\frac{d}{dr}(r^{-1}f^2w') = -4u^2\,e^{2v} \tag{2.9}$$

and

$$v' = \tfrac{1}{4}(rf^{-2}f'^2 - r^{-1}f^2w'^2) \tag{2.10}$$

while the equations (2.3) yield

$$u' + uv' = 0 \tag{2.11}$$

and

$$f^2uw' - rf(u' + uv') - u(f - rf') = 0 \tag{2.12}$$

where prime denotes derivative with respect to r.

Integration of equations (2.11) and (2.12) determines u as a function of v and w as a function of r and f. On substituting in (2.8) and (2.9) it is found that these two give the same equation for f, so that what at first sight appeared to be an overdetermined system is in fact not so. Equation (2.10) is then easily integrated. The final result is

$$\begin{aligned} &f = 4a^2r^2 + cr\log(pr) \qquad && w = b + rf^{-1} \\ &e^v = qr^{-1/4}f^{1/2} \qquad && u = aq^{-1}r^{1/4}f^{-1/2} \end{aligned} \tag{2.13}$$

where a, b, c, p and q are constants of integration with $p > 0$, $q > 0$. For the case in which $b \neq 0$ we can put $b = \pm 1$ without loss of generality, by simply rescaling the coordinates t, r and z and the remaining constants of integration.

Note that if $a = 0$, so that there is no electromagnetic field, we recover one of the three types of van Stockum (1937) vacuum metrics (see also Tipler 1974, Bonnor 1980). A thin shell source has been constructed for this purely gravitational case by Jordan and McCrea (1982) and the mass per unit coordinate length of z, calculated in accordance with a standard definition, was found to be $(1 + c)/4$ in the notation of the present paper. This would suggest that in the electrovac case the value of the constant c in (2.13) is a measure of the material mass of the presumed source, while clearly the value of a would be a measure of the charge. It would obviously be desirable to match the metric (2.1), (2.13) to a physically reasonable interior solution, but this has not yet been done.

3. Properties of the space–time

Since $g \equiv \det(g_{ij}) = -r^2 e^{2v}/f^2 < 0$ the signature of the metric is correct everywhere. The coefficient of $d\phi^2$ is given by

$$g_{\phi\phi} = f^{-1}(r^2 - f^2 w^2) = -r[4a^2b^2r + b^2c \log(pr) + 2b] \tag{3.1}$$

so that for sufficiently large values of r the vector field $\partial/\partial\phi$ is certainly time-like, which implies the existence of closed time-like curves. The existence of such curves for smaller values of r will depend on the positive or negative character of b, c and $(r - p^{-1})$. For the corresponding purely gravitational vacuum case ($a = 0$) considered by van Stockum, Tipler and Bonnor (Case II of Bonnor, equation (3b) of Tipler) where the vacuum exterior is matched to a dust interior solution, the constants b and c are negative and $r > p^{-1}$ so that closed time-like curves are excluded. However, even in the purely gravitational case (Case III of Bonnor, equation (3c) of Tipler) such lines do occur. Note that if $b = 0$, $\partial/\partial\phi$ is null for all values of r.

The Weyl tensor is of the Petrov type II, the degenerate principal null direction being the same as that of the electromagnetic field. Of the 14 curvature scalars (see, for instance, Campbell and Wainwright 1977) only two are non-zero, namely

$$I_1 \equiv C_{ab}{}^{cd}C_{cd}{}^{ab} = 3/(4q^4r^3)$$

and

$$I_3 \equiv C_{ab}{}^{cd}C_{cd}{}^{ef}C_{ef}{}^{ab} = -3/(16q^6r^{9/2}) \tag{3.2}$$

where C_{abcd} is the Weyl tensor. The remaining two pure Weyl scalars vanish identically together with all the pure Ricci and mixed scalars.

The three Killing vector fields $\partial/\partial t$, $\partial/\partial\phi$ and $\partial/\partial z$ span the Lie algebra of the complete isometry group. To see this we note that, since the curvature invariants I_1 and I_3 are functions of r, the orbits of the group are three dimensional. Furthermore, since the Weyl tensor is type II the principal tetrad is uniquely determined up to signs and hence the isotropy group is discrete i.e. zero dimensional. Thus the isometry group is three dimensional (see Kramer *et al* 1980, pp 59 and 114–5).

In one of the van Stockum space–times (Case I of Bonnor, equation (3a) of Tipler) there are time-like hypersurface-orthogonal (HSO) Killing fields which means that the space–time is, at least locally, static. For the metric (2.1), (2.13) with $b \neq 0$, if one considers a general Killing field of the form

$$\xi = n_0 \frac{\partial}{\partial t} + n_1 \frac{\partial}{\partial\phi} + n_2 \frac{\partial}{\partial z} \tag{3.3}$$

where n_0, n_1, n_2 are constants, it is found to be HSO if $n_1 = -n_0/b$, $n_2 = 0$ or if $n_0 = n_1 = 0$, $n_2 \neq 0$. In the former case it is null and in the latter space-like. For $b = 0$, ξ is HSO if either $n_1 \neq 0$, $n_0 = n_2 = 0$ or $n_2 \neq 0$, $n_0 = n_1 = 0$ and therefore ξ is again either null or space-like. Thus the metric (2.1), (2.13) yields a non-static stationary space–time.

4. Conclusion

The space–time presented above is an example of a strictly (i.e. non-static) stationary cylindrically symmetric electrovac field. The possibility of matching this solution to a physically reasonable source is being investigated.

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